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Decay of three-dimensional turbulence at high Reynolds numbers

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A modification of the Loitsyanskii integral is proposed as an invariant of three-dimensional decaying homogeneous and isotropic turbulence. As a result, the kinetic energy of a flow, generated in an infinitely large volume by the initial energy spectrum $E(k,t=0)$, which peaks in the vicinity of the wavenumber $k_0=1/\mathscr{L}(0)=O(1)$, $\mathscr{L}$ being an integral scale, decays with time in accordance with Kolmogorov's (1941) prediction: $\mathscr{K}=\frac{1}{2}\overline{u^2}\propto t^{-\gamma}$ with $\gamma=10/7$.

1. Background

The problem of the decay of three-dimensional turbulence, in which we are interested in this paper, can be formulated as follows. Consider the time evolution of an initial velocity field $\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0$ defined on an infinite space. The field $\boldsymbol{u}_0$ is a Gaussian random noise with the non-zero Fourier components $u_i(k)$ in the interval $k\geqslant k_i\approx 1/\mathscr{L}_0=O(1)$, where k is the wavenumber and $\mathscr{L}$ is an integral scale of turbulence, and it is assumed to have finite kinetic energy $\mathscr{K}(0)=\frac{1}{2}\overline{\boldsymbol{u}_0^2}=O(1)$. In other words, we consider an initial spectrum $E(k,0)$ rapidly approaching zero in the limit $k\rightarrow 0$ or $E(k\rightarrow 0,0)\ll k^n$ for any power n. In the absence of additional generation of turbulence, this field will decay with time. The decay process necessarily involves nonlinear interactions among wavenumbers. In this work we are interested in the decay process at intermediate times $\tau_0\ll t\ll\tau_\nu$ defined through $\tau_0\approx\mathscr{L}_0/u_0$ and $\tau_\nu\approx\mathscr{L}^2/\nu\rightarrow\infty$ when the viscosity $\nu\rightarrow 0$.

The energy balance for this flow, derived from the Navier–Stokes equations, is

$$\partial_t\mathscr{K}=-\epsilon=-\nu\overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2}<0. \tag{1}$$

The law of decay of kinetic energy $\mathscr{K}=\frac{1}{2}\overline{u^2}$ with time was first considered in a classic paper by Kolmogorov (1941). Kolmogorov's result $\mathscr{K}(t)\propto t^{-\gamma}$ with $\gamma=10/7$ has been challenged by Proudman & Reid (1954), Batchelor & Proudman (1956) and others (for details, see Monin & Yaglom 1975; Hinze 1975; Frisch 1995). While early experiments (see e.g. Batchelor 1953) yielded a power-law exponent of unity, the later, and more careful, experiments of Comte-Bellot & Corrsin (1966) yielded $\gamma\approx 1.25$–1.3; Bennet & Corrsin (1978) reported a $\gamma\approx 1.1$. An excellent and critical review of experimental data can be found in Skrbek & Stalp (2000). The results of numerical simulations (Lee & Reynolds 1995) also scatter quite substantially in the range $\gamma\approx 1.2$–1.67. The renormalization group calculation (Yakhot & Orszag 1986) based on an ϵ-expansion procedure gave $\gamma\approx 1.47$, reasonably close to Kolmogorov's value $\gamma=10/7$.

The decay of three-dimensional turbulence is one of the fundamental problems of turbulence theory. It is also one of the main benchmark cases used for calibration of various parameters entering semi-empirical models widely used in engineering (Launder & Spalding 1972). Thus, the magnitude of the exponent γ is quite important also for applications. The uncertainty in the measured or simulated value of γ can be explained by finite size effects and initial conditions: if the integral scale of turbulence $\mathscr{L}$ is approximately equal to the size of the system $L = \text{const}$, then we can approximate (1) as $\partial_t \mathscr{K} \approx -\mathscr{K}^{3/2}/L$ with the result: $\mathscr{K} \propto t^{-2}$. Indeed, in water experiments of a pull-through grid by van Doorn, White & Sreenivasan (1999), the exponent that was measured to be initially about 1.1 was found to cross over to a larger value for long times; similar results were found in the helium experiments of Smith *et al.* (1993). This basic result was also observed in various numerical simulations of Borue & Orszag (1995) and Biferale *et al.* (2003). Thus, the non-trivial time-dependence of $\mathscr{K}(t)$ is strongly tied to the simultaneous non-trivial time-dependence of the integral scale $\mathscr{L}(t) \ll L$. Since $\mathscr{L} \ll L$ grows with time, to achieve the asymptotic universal regime of turbulence decay, one needs really large-scale simulations. Our goal is to understand the asymptotic universal regime.

2. Kolmogorov's decay law and the limitations of the Loitsyanskii integral

Since Kolmogorov's derivation will be at the core of the theory to be developed below, let us first recall its main steps. Consider two points $\boldsymbol{x}$ and $\boldsymbol{x} + r\boldsymbol{i}$ in a flow, where $\boldsymbol{i}$ is the unit vector parallel to the x-axis, and define a correlation function $b_{rr} = \overline{u(\boldsymbol{x})u(\boldsymbol{x}+\boldsymbol{r})}$ where u is the x-component of the velocity field. In the limit $\nu \to 0$, the exact equation for b_{rr} is (Monin & Yaglom 1975; Landau & Lifshitz 1982)

$$\frac{\partial b_{rr}}{\partial t} = \frac{1}{6r^4}\frac{\partial}{\partial r} r^4 S_{3,0}(r), \tag{2}$$

with $S_{3,0} = \overline{(u(\boldsymbol{x}+\boldsymbol{r}) - u(\boldsymbol{x}))^3}$. Multiplying this equation by r^4 and integrating over r from $r = 0$ to $r = \infty$, we obtain

$$\frac{\partial}{\partial t}\int_0^\infty r^4 b_{rr}\,\mathrm{d}r = \tfrac{1}{6} r^4 S_{3,0}(r)|_0^\infty. \tag{3}$$

If we assume (following Loitsyanski 1939) that as $r \to \infty$, the product $r^4 S_{3,0}(r) \to 0$, we have the result that $I = \int_0^\infty r^4 b_{rr}\,\mathrm{d}r = \text{const}$. The integral is estimated readily (Kolmogorov 1941): since as $r \to \infty$ the correlation function b_{rr} rapidly tends to zero, the integral is approximately equal to $\mathscr{K}\mathscr{L}^5$ where $\mathscr{L}$ is the integral scale beyond which the correlation function rapidly disappears. Since $\mathscr{L} \approx u_{rms} t$, we readily obtain $\mathscr{K} \propto t^{-10/7}$ and $\mathscr{L} \propto t^{2/7}$. This is Kolmogorov's law of turbulence decay. Later, Landau & Lifshitz (1982) showed that the Loitsyanskii invariant $I = \text{const}$ can be obtained directly from conservation of angular momentum of the entire flow.

This beautiful derivation was analysed by Proudman & Reid (1956) and Batchelor & Proudman (1956), and was shown to be incorrect (for a review, see Monin & Yaglom 1975; Hinze 1975; Frisch 1995). To demonstrate its breakdown, we recall that

$$I = \int_0^\infty r^4 b_{rr}\,\mathrm{d}r = 2\int_0^\infty r^4\,\mathrm{d}r \int_0^\infty E(k)\cos(kr)\,\mathrm{d}k = \frac{\partial^4 E(k=0)}{\partial k^4}. \tag{4}$$

Thus, the statement $I = \text{const}$ is equivalent to the statement that $\partial_t \partial^4 E(k=0,t)/\partial k^4 = 0$. That this is not so can be illustrated as follows. We can utilize the scale separation $(k\mathscr{L}(t) \to 0; \omega \to 0)$, ω being the frequency, and using the multiscale or renormalization group methods (Forster, Nelson & Stephen 1977; Yakhot & Orszag

1986; Frisch 1995) derive a linear equation for time evolution of the energy spectrum as

$$\frac{\partial E(k \to 0)}{\partial t} = ak^4 \mathscr{L}^4 u_{rms}^3 - bk^2 u_{rms} \mathscr{L} E(k \to 0), \tag{5}$$

subject to the initial condition $E_0 = E(k, 0)$. In the limit $k \to 0$, the nonlinear advective contribution to (5) disappears due to Galilean invariance (Forster *et al.* 1977). The equation (5) is the outcome of a multiscale expansion in powers of dimensionless parameter $k\mathscr{L}$, so that $a = \sum_{n=0}^{\infty} A_n (k\mathscr{L})^{2n}$ and $b = \sum_{n=0}^{\infty} B_n (k\mathscr{L})^{2n}$. As we are interested in the limit $k\mathscr{L} \to 0$, we may neglect all high-order contributions and treat the coefficients a and b as constant geometric coefficients. It will become clear that the most (and only) important property of (5) for the theory presented below is that if the small-scale velocity field $\boldsymbol{u}$ is statistically isotropic and homogeneous, the parameters $a > 0$ and $b > 0$. Equation (5) governs the time evolution of the small-wavenumber asymptotics ($k \to 0$) of the energy spectrum $E(k)$ with the first term in the right-hand side describing the eddy noise (back scattering), transferring some of the energy toward large scales. The second contribution, leading to energy dissipation, is responsible for the so-called eddy viscosity effects. Combining (4) and (5), we conclude that the Loitsyanskii integral is not an invariant after all! As already pointed out, this conclusion was first reached by Proudman & Reid (1954) and Batchelor & Proudman (1956). (For the most recent review of the subject, see also Skrbek & Stalp 2000.)

The solution to equation (5) demonstrates the well-known effect of back scattering: while the total energy in the system decays, the initially depleted long-wave part of the energy spectrum

$$E(k \to 0, t) \approx ak^4 \exp\left(-bk^2 \int \mathscr{L} u_{rms}\right) \int_0^t \mathscr{L}^4 u_{rms}^3 \exp\left(bk^2 \int \mathscr{L} u_{rms}\right) + E(k \to 0, 0)$$

steadily grows with time, so that $\partial_t \partial^4 E(k=0)/\partial k^4 \approx \mathscr{L}^4 u_{3ms}^3 > 0$. As we will see later, as time $t \to \infty$, this rate of growth tends to zero as a negative power of t. Here we are interested in the case of the initial spectrum $E(0, 0) = 0$ and it follows from the above solution that $E(k \to 0, t) \propto k^4$. Thus $\partial_t I(t) > 0$, so that by repeating Kolmogorov's argument, we obtain $\mathscr{K}(t) \propto t^\gamma$ with $\gamma \geqslant -10/7$. To obtain a more solid result, we have to modify the theory.

3. The new lengthscale and the principal result

Introducing dimensionless variables $T = tu_o/\mathscr{L}_0$, $U = u/u_0$, $K = k\mathscr{L}_0$, $l = \mathscr{L}/\mathscr{L}_0$ and $e(k) = E(k)/\mathscr{L}_0 u_0^2$, equation (5) becomes

$$\frac{\partial e(K \to 0)}{\partial T} = aK^4 l^4 U^3 - bK^2 U l e(K \to 0).$$

We see that in the dimensionless variables, the initial parameters $\mathscr{L}_0$, u_0 disappear from the equations of motion of the large-scale velocity fluctuations, and thus cannot enter the finial result. This means that in an infinite flow with viscosity $\nu \to 0$ the long-time asymptotics $T \gg 1$ can involve only the dynamically determined large lengthscale $\mathscr{L}(t) \approx u_{rms}(t) t$ which is similar to the one used by Kolmogorov (1941).

Let us multiply equation (2) by r^4 and integrate over r in the interval $0 \leqslant r \leqslant \mathscr{L}_Y(t)$ to obtain

$$\int_0^{\mathscr{L}_Y(t)} \frac{\partial}{\partial t} r^4 b_{rr}(r)\,\mathrm{d}r = \frac{\partial}{\partial t} \int_0^{\mathscr{L}_Y(t)} r^4 b_{rr}(r)\,\mathrm{d}r - \frac{\partial \mathscr{L}_Y}{\partial t} \mathscr{L}_Y^4 b_{rr}(\mathscr{L}_Y) = \tfrac{1}{6} \mathscr{L}_Y^4 S_{3,0}(\mathscr{L}_Y).$$

Defining the time-dependent 'integral' scale as

$$\mathscr{L}_Y(t) = -\int_0^t \mathrm{d}\tau \frac{S_{3,0}(\mathscr{L}_Y(\tau))}{6b_{rr}(\mathscr{L}_Y(\tau))}, \tag{6}$$

we see that the integral

$$I_Y = \int_0^{\mathscr{L}_Y(t)} r^4 b_{rr}\,\mathrm{d}r = \text{const}, \tag{7}$$

is an invariant of decaying turbulence. In the inertial range, where $S_3(\mathscr{L}_Y) \propto -\mathscr{L}_Y < 0$, equation (6) does not have solutions. Thus, the scale $\mathscr{L}_Y(t)$, defined by (6), can pertain only to the upper end of the inertial range where $S_{3,0}(r)$ tends toward zero. The existence of this scale has been demonstrated in the measurements of Kurien & Sreenivasan (2001) and the simulations of Gotoh & Nakano (2003). The relation (6) leads to the important estimate that $\mathscr{L}_Y \approx u_{rms}t$.

To evaluate the integral in (7), let us define $E^<(k)$ and $E^>(k)$ corresponding to $k \leqslant 1/\mathscr{L}_Y$ and $k \geqslant 1/\mathscr{L}_Y$, respectively, and, introduce the integral wavenumber $k_Y = 1/\mathscr{L}_Y$. Let us further demand the continuity of the energy spectrum at the integral scale $E^<(k_Y) = E^>(k_Y)$. Since we have shown that $E^<(k)$ involves the initial lengthscale only in the combination $K = k\mathscr{L}_0$, we conclude that at the top of the universal ('inertial') range $e^> = e^>(\mathscr{L}_0/\mathscr{L}_Y(t))$. Since on the scales $r > \mathscr{L}_Y$ the correlation function rapidly decreases, the integral (7) is proportional to $\mathscr{L}_Y(t)^5 u_{rms}^2$. At the same time, it follows from (4) and (5) that the same integral has the form

$$\begin{aligned} I_Y = \int_0^{\mathscr{L}_Y} r^4 b_{rr}\,\mathrm{d}r &= 2\int_0^{\mathscr{L}_Y} r^4\,\mathrm{d}r \int_0^{\infty} E(k)\cos(kr)\,\mathrm{d}k \\ &= \int_0^{\infty} \frac{\partial^4 E(k)}{\partial k^4}\left(\frac{\sin(k\mathscr{L}_Y)}{k} - \mathscr{L}_Y\right)\mathrm{d}k = I_1 + I_2. \end{aligned} \tag{8}$$

Due to the strongly oscillating factor $k\mathscr{L}_Y \gg 1$, the main contribution to the first integral I_1 on the right-hand side of (8) comes from the interval $0 < k < 1/\mathscr{L}_Y$. Using equation (5) for the energy spectrum in this interval, gives $I_1 = O(u_{rms}^3 \mathscr{L}_Y^4 t)$. Since in the limit $k \to 0$, the energy spectrum $E(k) \propto k^4$, the second integral $I_2 = -\mathscr{L}_Y(\partial^3 E(k)/\partial k^3)|_0^\infty = 0$. On the other hand, the above estimate gave $I_Y \approx u_{rms}^2 \mathscr{L}_Y^5 > 0$. Thus, we have $u_{rms}^3 \mathscr{L}_Y^4 t \approx u_{rms}^2 \mathscr{L}_Y^5$ leading to the relation $\mathscr{L}_Y \approx u_{rms}t$, similar to the one obtained above. Using this, we can repeat Kolmogorov's argument ($I_Y \approx \mathscr{L}_Y^5 u_{rms}^2 = \text{const}$; $\mathscr{L}_Y \approx u_{rms}t$) and obtain the law of turbulence decay: that, for long times, $T \gg 1$ ($t \gg t_0$), the kinetic energy $\mathscr{K}(t)/\mathscr{K}(0) \propto (t/t_0)^{-10/7}$ where $t_0 \approx \mathscr{L}_0/u_0$.

4. Conclusions

In the past, Kolmogorov's decay law has been regarded as inapplicable to the decay problem because its original derivation was based on the existence of the Loitsyanskii integral. The main result of this work is the following. We have discovered a new integral lengthscale $\mathscr{L}_Y(t)$ defined by (6) and related to it the integral (7), which is an invariant (integral of motion) of decaying high-Reynolds-number turbulence. The lengthscale is characterized by the condition that the third-order structure function $S_{3,0}(\mathscr{L}(t))$ rapidly reaches zero around it. As a result, in an infinitely large system the Kolmogorov decay law, $\mathscr{K}(t) \propto t^{-\gamma}$ with $\gamma = 10/7$, is recovered. Its validity does not depend on the limitations of the Loitsyanskii integral.

The method developed above fails when applied to the problem of decaying turbulence governed by the one-dimensional Burgers equation, generated by the random initial conditions with $E(k=0,0)=0$, we are interested in here. In this case, the integral $I=\int_{-\infty}^{\infty} b_{rr}\,\mathrm{d}r=E(k=0,t)=0$. As a result the estimate $I\approx u^2\mathscr{L}=\mathrm{const}\neq 0$ is incorrect. The integral scale in this case is given by the $\mathscr{L}\approx u_{rms}t$, as in the case of the Navier–Stokes turbulence, considered above, but an additional equation for $\mathscr{L}(t)$ is needed. This problem was analysed in detail for the class of initial conditions $E(k,0)=k^n\phi(k)$ by Gurbatov *et al.* (1997).

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